The Tetrahedron: Elementary Form, Quantum Geometry and the Question of Inside and Outside

Reading instructions: Read this text several times, examining and researching just one aspect in more detail each time. Then return to it after some time has passed and assess your own understanding in light of your now-changed perspective on the subject.

1. The ancient archetype

In Plato's **Timaeus**, the tetrahedron is the elementary geometric form of fire. Together with the cube (earth), the octahedron (air) and the icosahedron (water), it forms four of the five regular solids from which Plato posits the visible world to be constructed – based on the argument that the cosmos, as a perfect work, must possess the highest possible degree of symmetry *(Plato, Timaeus, Chapters 20–21, 53c–56c; cf. Wikipedia 'Timaeus' and 'Platonic solids')*. The tetrahedron is not chosen arbitrarily, but is the simplest of the five solids – the simplest three-dimensional figure of all that can be formed from four triangular faces.

This fundamental role of the tetrahedron is reflected in the Pythagorean numerology: the *Tetraktys* (a triangle formed by ten points arranged in four rows) was regarded as the geometric origin of the dimensions – point, line, plane, solid – whereby the fourth row can, for the first time, form a tetrahedron *(article 'Tetraktys', Wikipedia / AnthroWiki)*. Blavatsky links this structure to the Tetragrammaton: the perfect triangle becomes the Tetraktys, the 'dual symbol' of the four-lettered name of God *(The Secret Doctrine, Vol. II)*. In Steiner's work, the tetrahedron reappears elsewhere – as a cosmically determined fundamental form, both in crystal formation (as the preserved interplay of Mars, Venus and Mercury from the ancient lunar state) and, on a large scale, as the postulated fundamental structure of the Earth itself *(GA 354, twelfth lecture; AnthroWiki 'Tetrahedral Structure of the Earth')*.

2. The tetrahedron in quantum gravity

> *Question: In quantum physics, which is predominantly based on higher mathematics, does a reference to the Platonic solids – in particular the 'fiery' tetrahedron – actually play a discernible role, or is this merely a superficial analogy?*

In *loop quantum gravity* (LQG – one of several competing approaches to unifying space, time and gravity with the principles of quantum mechanics), space is described not as a continuum but as *a web of spins*: a graph consisting of vertices and edges, whose edges are assigned half-integer values (spin quantum numbers – a parameter adopted from the quantum mechanics of elementary particles). The *volume operator* (the mathematical tool used in quantum theory to assign a measurable value to a quantity – in this case, the volume of a region of space) acts on these nodes; its spectrum (the set of possible measured values) has been shown to be discrete, not continuous as in ' ' *(C. Rovelli, L. Smolin, 'Discreteness of Area and Volume in Quantum Gravity', Nuclear Physics B 442, 1995, preprint November 1994)*.

The simplest, 'most elementary' type of node in this network – one with four edges – corresponds mathematically exactly to a tetrahedron: Its four faces are represented by angular momentum operators (mathematical objects that describe rotational properties in quantum mechanics), which couple to form a so-called spin singlet (a state in which the individual angular momenta cancel each other out to give a total value of zero). This *quantum tetrahedron* is regarded as the elementary, indivisible volumetric quantum of space at the Planck scale (the smallest physically meaningful unit of length, approximately 10^{-35} metres) *(A. Barbieri, 'Quantum tetrahedra and simplicial spin networks', Nuclear Physics B 518, 1998; J. Baez, J. Barrett, 'The Quantum Tetrahedron in 3 and 4 Dimensions', Adv. Theor. Math. Phys. 3, 1999; later generalisation to arbitrary polyhedra: E. Bianchi, P. Doná, S. Speziale, 'Polyhedra in Loop Quantum Gravity', 2011)*.

It is worth noting that, independently of LQG, a second, methodologically entirely different approach – *causal dynamical triangulations* (a method that discretises spacetime using path integral methods rather than canonical quantisation) – also arrives at tetrahedral-shaped fundamental building blocks of spacetime *(R. Loll et al.; see the *Spektrum* article 'Loop Quantum Gravity: A Spacetime of Loops')*. This suggests a mathematical necessity rather than a deliberately sought symbolism: the tetrahedron is the simplest possible unit of decomposition of three-dimensional space, regardless of the chosen physical approach.

Whether this is a mere mathematical coincidence – that the simplest form repeatedly appears as the simplest form – or an indication of a deeper structural necessity of three-dimensional space, which Plato may have intuitively grasped even without today's apparatus of proof, cannot be determined and should be left open.

3. Evolutionary, revolutionary or incidental?

> *Question: What significance does this finding have for scientific thought – is its character more one of continuation (evolutionary), upheaval (revolutionary) or, ultimately, incidental (casual)?*

A distinction must be made between two levels. Methodologically, the path to this result is evolutionary: The discreteness of spatial geometry was not a new fundamental assumption, but rather the consequence of consistently applying a method established for decades – non-perturbative quantisation (a method of converting classical physical quantities into quantum mechanical operators without assuming approximations based on small perturbations) – to the General Theory of Relativity. This is the continuation of an established research programme, only – radically – taken to its logical conclusion: the metric of space and time itself.

Should this *result* ever be confirmed, however, it would certainly be revolutionary in nature: the dismantling of the notion of a continuous space-time continuum in favour of a granular, discrete structure. However, the theory remains experimentally unconfirmed to this day, is scientifically controversial, and is merely one of several competing approaches (string theory, causal dynamic triangulations, asymptotic safety, causal sets) – so a paradigm shift has not (yet) taken place, but rather a candidate for one has emerged.

The special role of the tetrahedron itself, by contrast, should be regarded as rather incidental: it appears not because any particular quality is attributed to it, but because it is the mathematically most economical solution – a by-product of the construction, not a sought-after goal.

4. Quantisation as the 'rhythmisation' of space

> *Question: Can quantisation be seen as a kind of 'rhythmisation' of what was previously conceived as a pure continuum?*

Technically speaking, quantisation (the transformation of classical quantities into operators) does not necessarily entail discreteness; whether a spectrum turns out to be discrete or continuous depends on whether the underlying system is compact or closed (as with bound orbits in an atom or the closed loops of the LQG) or open (as with a free particle in infinite space, whose momentum spectrum remains continuous even after quantisation).

From a historical linguistic perspective, this gives rise to an illuminating linguistic device: Aristotle reports that Democritus defined his atoms by three properties – *rhythmos, diathige, trope* (Metaphysics I, 4, 985b) – which he translated as 'form', 'arrangement' and 'position'. 'Rhythmos' therefore originally did not mean periodic recurrence in time (the modern musical sense), but rather **shape, form, structure** – the specific character by which one atom differs from another. It was only later that the term narrowed to its temporal-periodic meaning. If one understands 'rhythmisation' in this older, Democritian sense, the formulation captures the process quite accurately: quantisation gives a continuum a countable form – a *'rhythmos'* in the original sense of the word – where previously there was an undifferentiated whole. In this picture, the tetrahedron would be precisely such a 'rhythmos': the minimal form imprinted upon space, through which it first becomes countable and structured.

5. A concluding image: the medial tetrahedron

If the four face centres of a tetrahedron are marked and connected, a second tetrahedron resting on its apex is formed. Geometrically, this is a point reflection of the original solid about their common centre of gravity, with a reduction factor of 1/3. It follows that: The volume of the smaller solid is not one third, but $(1/3)^3 = \mathbf{1/27}$ of the original volume; its entire shape and position are derived entirely from the four points of contact of the larger solid, without any independent construction rule of its own.

> *Question: Should the process of creating this figure be conceived from the outside in – or from the inside out?*

From a purely mathematical perspective, the underlying mapping is reversible: just as the smaller body can be derived from the larger one, the larger body can conversely be recovered from the smaller one by reflection with a factor of –3. However, the *construction process*, as formulated linguistically – the determination of the centres of the faces – necessarily presupposes the prior, complete existence of the larger body and thus describes a movement from the outside in: the encompassing whole as the condition from which the derived form emerges. At the same time, since the transformation is reversible, the smaller body always

already carries within it, in a virtual sense, the complete blueprint of the larger one – an interpretation that would correspond more to a movement from the inside out: the encompassing whole as that which is already laid down within the smaller, and which could unfold from it.

Thus, the classical tension between involution (the larger descending, condensing and individuating itself into form) and evolution (the smaller unfolding what is already latent in its seed) becomes apparent in geometry itself – not as an either/or, but as a single movement which, depending on the chosen starting point of observation, manifests as descent or as unfolding.

A parallel from the Vedantic tradition

In his commentary on the Bhagavad Gita (**God Talks with Arjuna**, Chapter 15), Paramahansa Yogananda interprets the ancient Vedic image of the Ashvattha tree, whose roots lie above and whose branches lie below: The visible, material world is said to be the downward-growing, inverted manifestation of a reality whose true origin – the 'root' – lies in the spiritual realm and remains beyond the reach of sensory perception. He applies the same principle of inversion to the story of creation: human beings in their natural state live, as it were, upside down, with their stream of perception and energy perpetually directed outwards and downwards; Knowledge of the spiritual world requires the complete reversal of this flow, turning it inwards and upwards *(P. Yogananda, *God Talks with Arjuna: The Bhagavad Gita*, commentary on Chapter 15; cf. also *Autobiography of a Yogi*, Chapter 16)*.

This reversal as a principle of knowledge touches upon the open question concerning the medial tetrahedron: there, too, it could be shown that one and the same geometric relationship – depending on which pole from which one interprets it – appears either as descent (involution, the outside determining the inside) or as unfolding (evolution, the inside striving outwards). Yogananda's image adds a third, complementary nuance to this: the 'correct' direction of gaze itself is not given neutrally, but must be actively attained – for him, knowledge of the spiritual world is not merely a change of perspective in observation, but a movement of consciousness that must be reversed in the literal sense.

Final thought from the questioner

The space referred to here is merely one aspect, one 'level', of a more comprehensive space, within which the human-soul components and, likewise, the spiritual core of origin of every human entity are also to be found. It is just as real as the thought of it is real – even if it cannot be perceived by the senses. Quantum physics and its theorems offer significant scope for cautious analogies; they help the ever-in-progress thinker of theosophy to refine his approaches and endeavours. To put it another way: a 'granular' conception of space is illuminated or permeated in its 'interstices' by that which may be assumed to be a larger, more comprehensive space. Paradoxically, the smaller, inverted tetrahedron can be seen as representing the larger space.

Selected references

- *Plato, Timaeus, Chapters 20–21 (53c–56c)*
- *Aristotle, Metaphysics I, 4, 985b (on Democritus)*
- *H. P. Blavatsky, The Secret Doctrine, Vol. II*
- *Rudolf Steiner, GA 354, twelfth lecture, Dornach, 18 September 1924*
- *C. Rovelli, L. Smolin: 'Discreteness of Area and Volume in Quantum Gravity', Nuclear Physics B 442 (1995), 593–619*
- *A. Barbieri: 'Quantum tetrahedra and simplicial spin networks', Nuclear Physics B 518 (1998), 714–728*
- *J. Baez, J. Barrett: The Quantum Tetrahedron in 3 and 4 Dimensions, Adv. Theor. Math. Phys. 3 (1999), 815–850*
- *E. Bianchi, P. Doná, S. Speziale: Polyhedra in Loop Quantum Gravity (2011)*
- *P. Yogananda: God Talks with Arjuna: The Bhagavad Gita, commentary on Chapter 15; and Autobiography of a Yogi, Chapter 16*